GPCO 453: Quantitative Methods I

Sec 02: Time Preferences

Shane Xinyang Xuan[1]
ShaneXuan.com

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[1]Department of Political Science, UC San Diego, 9500 Gilman Drive #0521.

Contact Information

Shane Xinyang Xuan

xxuan@ucsd.edu

The teaching staff is a team!

Professor Garg	Tu	1300-1500 (RBC 1303)
Shane Xuan	M	1100-1200 (SSB 332)
	M	1530-1630 (SSB 332)
Joanna Valle-luna	Tu	1700-1800 (RBC 3131)
	Th	1300-1400 (RBC 3131)
Daniel Rust	F	1100-1230 (RBC 3213)

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- When $n = 1$, we have

$$A = P\left[1 + \frac{r}{1}\right]^{1\times t} \tag{2}$$

$$= P(1 + r)^{t} \tag{3}$$

Geometric Series

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- Subtracting (5) from (4),

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- It follows that

$$S_n = \frac{1 - r^{n+1}}{1 - r} \tag{8}$$

Geometric Series (2)

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- In general,

$$S_n = a_0 \frac{1 - r^{n+1}}{1 - r}, \; r \neq 1 \quad (12)$$

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- We are trying to compute

$$
\begin{aligned}
&1000(1+0.02)^{20} + 1000(1+0.02)^{19} + \ldots + 1000(1+0.02)^{11} \\
&= 1000(1.02)^{11} \underbrace{\left(1 + 1.02 + 1.02^2 + \ldots + 1.02^9\right)}_{\sum_{k=0}^{k=9} a_k,\, a_0=1}
\end{aligned}
\tag{13}
$$

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- Consider $S_n = \sum_{k=0}^{n} a_k = a_0 \frac{1-r^{n+1}}{1-r}$ again:

$$S_n = \frac{1 - 1.02^{9+1}}{1 - 1.02} = 10.94972 \tag{15}$$

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- Hence,

$$FV = 1000(1.02)^{11}(10.94972) = 13614.6 \qquad (16)$$

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$$PV = A\left[1 + \frac{1-(1+r)^{-(n-1)}}{r}\right] \tag{18}$$

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  Using similar logic, $PV^\dagger = \frac{77217}{(1+0.05)^2} \approx 70038$

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$$59673 = PV\left[1 + 0.08\right]^4 \tag{19}$$

$$PV \approx 43861 \tag{20}$$

Practice Problem 2

- Find the present value of $\$10000$ received 3 years from now assuming 1st year interest rate is 3%, 2nd year interest rate is 5%, and 3rd year interest rate is 1%.

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  Recall that

$$
\begin{aligned}
PV &= FV(1+r)^{-t} && (21)\\
&= \underbrace{10000}_{\text{FV}}\underbrace{(1+0.03)^{-1}\underbrace{(1+0.05)^{-1}\underbrace{(1+0.01)^{-1}}_{\text{3rd year}}}_{\text{2nd year}}}_{\text{1st year}} && (22)\\
&\approx 9155 && (23)
\end{aligned}
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Practice Problem 3

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- Same as before, but the 1st \$1000 payment starts at the end of the 5th year.
  - $25000 - \underbrace{\frac{1000}{0.04}\left[1 - \frac{1}{(1+0.04)^4}\right]}_{\text{annuity for 4 years}} \approx 21370$

Practice Problem 4

- The proprietors of a hotel secured two loans from a local bank: one for $\$800000$ due in 3 years and one for $\$1500000$ due in 6 years. Both loans are at an interest rate of 10% per year. The bank agrees to allow the two loans be consolidated into one loan payable in 5 years at the same interest rate. How much will the proprietors have to pay the bank at the end of year 5?

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  - $PV = PV^{(1)} + PV^{(2)} \approx 1447762.7$

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  - $PV = PV^{(1)} + PV^{(2)} \approx 1447762.7$
  - $FV = PV(1+r)^t = 1447762.7 \times (1.1)^5 \approx 2331636$