GPCO 453: Quantitative Methods I

Sec 05: Probability, continued

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Contact Information

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The teaching staff is a team!

Professor Garg	Tu	1300-1500 (RBC 1303)
Shane Xuan	M	1100-1200 (SSB 332)
	M	1530-1630 (SSB 332)
Joanna Valle-luna	Tu	1700-1800 (RBC 3131)
	Th	1300-1400 (RBC 3131)
Daniel Rust	F	1100-1230 (RBC 3213)

Roadmap

In this section, we cover the basics for probability:

- Basic relationship of probability

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- Independence and mutual exclusivity
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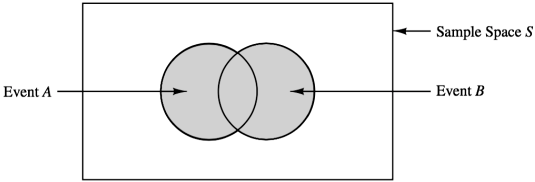
- Z-score
- Probability distribution

Basic Relationships of Probability

- Union (denoted by $\cup$): The union of A and B is the event containing **all** sample points belonging to A or B or both.



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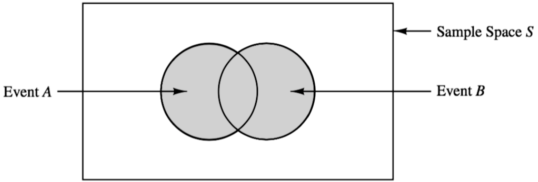
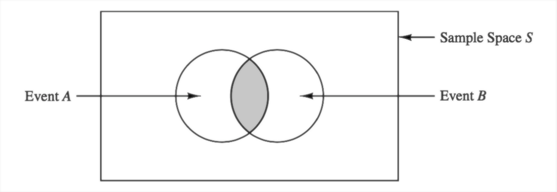
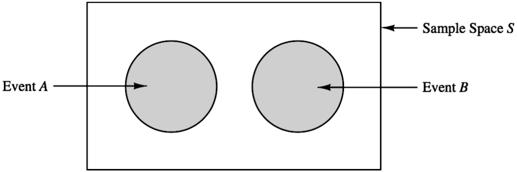


- Intersection (denoted by $\cap$): The intersection of A and B is the event containing the sample points belonging to **both** A and B.



Independence and Mutual Exclusion

- Events A and B are mutually exclusive if, when one event occurs, the other **cannot** occur; $\Pr(A \cup B) = \Pr(A) + \Pr(B)$



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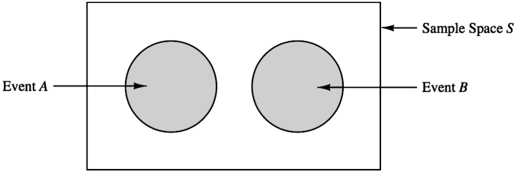
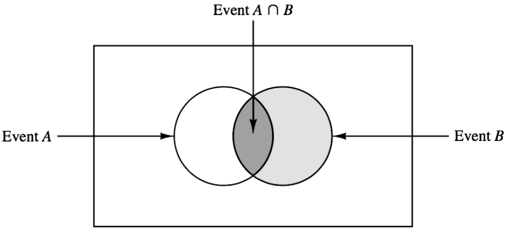


- Two events A and B are independent if

$$\Pr(A|B) = \Pr(A) \quad (1)$$
$$\Pr(B|A) = \Pr(B) \quad (2)$$

More on Conditional Probability



$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{conditional probability} \qquad (3)$$

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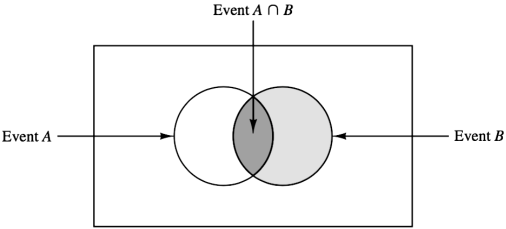

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- Multiplication law

$$\Pr(A \cap B) = \Pr(B)\Pr(A|B) \quad \text{general case} \quad (5)$$

$$\Pr(A \cap B) = \Pr(B)\Pr(A) \quad \text{independent events} \quad (6)$$

Bayes' Theorem (Two-Event Case)

$$\Pr(A_1|B) = \frac{\Pr(A_1)\Pr(B|A_1)}{\Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2)} \tag{7}$$

$$\Pr(A_2|B) = \frac{\Pr(A_2)\Pr(B|A_2)}{\Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2)} \tag{8}$$

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- Important to note that

$$\Pr(A_1) + \Pr(A_2) = 1 \tag{9}$$

$$\Pr(A_1 \cap B) + \Pr(A_2 \cap B) = \Pr(B) \tag{10}$$

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$$\Pr(\text{putt}|\text{par}) = \frac{\Pr(\text{putt})\Pr(\text{par}|\text{putt})}{\Pr(\text{putt})\Pr(\text{par}|\text{putt}) + \Pr(\neg\text{putt})\Pr(\text{par}|\neg\text{putt})}$$
$$= \frac{.61 \times .64}{.61 \times .64 + .39 \times .203} \approx .831 \qquad (11)$$

Z-score

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$$Z \equiv \frac{X - \mu_X}{\sigma_X}, \tag{12}$$

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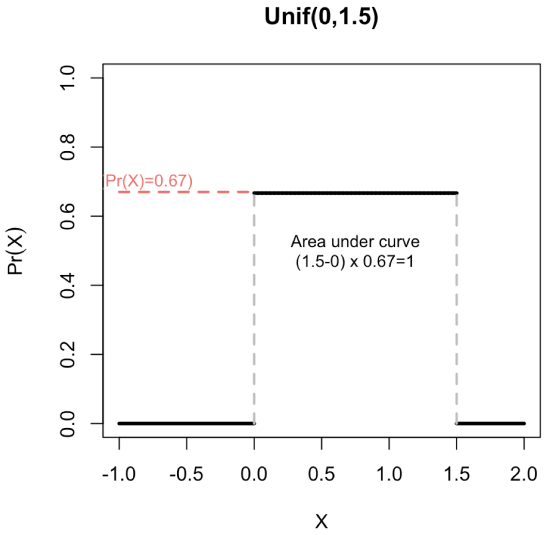
- **Outliers** are defined as observations having a Z-score below -3 or more than 3.

Uniform Distribution



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- Area under the curve is $\underbrace{(5)}_{\text{length}} \underbrace{\left(\frac{1}{30}\right)}_{\text{height}} = \frac{1}{6}$

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Binomial Distribution: Example

- A university found that 20% of its students withdraw without completing the introductory statistics course. Assume that 20 students registered for the course. Compute the probability that more than 2 will withdraw.

$$
\begin{aligned}
\Pr(X > 2) &= 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2) \\
&= 1 - \binom{20}{0} .2^0 (1 - .2)^{20-0} - \binom{20}{1} .2^1 (.8)^{20-1} - \binom{20}{2} .2^2 (.8)^{20-2}
\end{aligned}
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- Compute the expected number of withdrawals.

$$\mathbb{E}[X] = np = 20 \times .2 = 4 \tag{18}$$

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- Both mean and variance are μ
- During the period of time that a local university takes phone-in registrations, calls come in at the rate of one every two minutes. What is the probability of three calls in five minutes?

$$\Pr(\text{three calls in five minutes}) = \frac{(2.5)^3 e^{-2.5}}{3!}$$

Appendix: Code for Uniform Distribution

```
x <- seq(0,1.5,length=5000)
unif <- dunif(x, min=0, max=1.5)
plot(y=unif, x=x, type="l", lwd=3, xlim=c(-1,2), ylim=c(0,1),
     main="Unif(0,1.5)", ylab=expression(Pr(X)), xlab='X')
segments(x0=-1, x1=0, y0=0, y1=0, lwd=3)
segments(x0=1.5, x1=2, y0=0, y1=0, lwd=3)
segments(x0=0, x1=0, y0=0, y1=0.67, lty=2, lwd=2, col="gray")
segments(x0=1.5, x1=1.5, y0=0, y1=0.67, lty=2, lwd=2, col="gray"
segments(x0=-1, x1=0, y0=0.67, y1=0.67, lty=2, lwd=2, col="india
text(-.8,.7,"(Pr(X)=0.67)", col="indianred1", cex=.85)
text(.75,.5,"Area under curve\n(1.5-0) x 0.67=1", cex=.85)
```