GPCO 453: Quantitative Methods I

Sec 04: Introduction to Probability

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Contact Information

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The teaching staff is a team!

Professor Garg	Tu	1300-1500 (RBC 1303)
Shane Xuan	M	1100-1200 (SSB 332)
	M	1530-1630 (SSB 332)
Joanna Valle-luna	Tu	1700-1800 (RBC 3131)
	Th	1300-1400 (RBC 3131)
Daniel Rust	F	1100-1230 (RBC 3213)

Roadmap

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- Covariance

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- Variance
- Expectation

Central Tendency: Expectation

- If the probability distribution of X admits a probability density function $f(x)$, then the expected value can be computed as

$$\mathbb{E}[X] = \begin{cases} \sum_{x \in \mathcal{X}} x f(x) & \text{if discrete} \\ \int_{x \in \mathcal{X}} x f(x) dx & \text{if continuous} \end{cases} \tag{1}$$

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- Expectation can be viewed as a weighted mean.

Expectation: Example

The County of San Diego has conducted a food facility inspection search of 300 restaurants. Each restaurant received a rating on a 3-point scale on typical meal price (in columns) and quality (in rows).

Table: Food Facility Inspection Results

	1	**2**	**3**	**Total**
1	42	39	3	84
2	33	63	54	150
3	3	15	48	66
Total	78	117	105	300

a.) Develop a bivariate probability distribution for quality (x) and meal price (y) of a randomly selected restaurant in San Diego.

b.) Compute the expected value for quality rating, x.

Variability: Variance

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- We define the variance to be the expected distance from X to μ_X:

$$\text{Var}(X) \equiv \mathbb{E}[(X - \mu_X)^2] \tag{2}$$

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- If X is above its mean when Y is also above its mean and vice versa, then the covariance will be **positive**.

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- Note that $\rho_{xy} \in [-1, 1]$

Sample Covariance: Example

Consider the table below:

x_i	6	12	13	15
y_i	5	6	8	1

Compute covariance and correlation coefficient. Recall that

$$\sigma_{xy} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{n-1} \tag{5}$$

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} \tag{6}$$